Duality-Invariant Non-linear Electrodynamics and Stress Tensor Flows

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Christian Ferko

Center for Quantum Mathematics and Physics (QMAP)
Department of Physics & Astronomy, UC Davis

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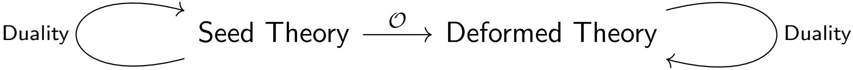
Motivation: symmetries and deformations.

We can often learn a great deal by beginning with a well-understood, tractable system and then *deforming* it in some controlled way.

By "controlled" we typically mean that the deformation preserves some desirable property of the system, such as integrability or supersymmetry.



In this talk, the desirable property we consider is **electric-magnetic duality invariance**, and deformations which preserve it as in this diagram.

Dualities relate different theories.

It is natural to consider dualities because they are ubiquitous in physics.

A

- **Holographic duality** relates a d-dimensional gravitational theory, or string theory, to a conformal field theory in $(d-1)$ dimensions.
- **Particle-vortex duality** in $3d$ gauge theory relates two theories with different Lagrangians (e.g. Abelian Higgs and XY models) which describe the same physics in terms of different degrees of freedom.

B

- **T-duality** relates string theory on a target spacetime with a circle of size R to string theory on a spacetime with a circle of size $\frac{\alpha'}{R}$.
- **Strong-weak dualities**, like the S-duality of type IIB string theory or Montonen-Olive duality, typically relate a theory at some value τ of a complex coupling parameter to a theory at value $-\frac{1}{\tau}$ of this coupling.

Both categories involve two different descriptions of the same system.

Electric-magnetic duality.

Generically, a duality relates *different* theories: either (***A***) theories with different fundamental degrees of freedom, or (***B***) with the same DoF but with different values of certain coupling constants which define the theory.

An example of type (***B***), and our main focus here, is **electric-magnetic duality**. In the presence of both electric sources j_{e}^{μ} and magnetic sources j_{m}^{μ}, the equations of motion for the Maxwell theory are

$$\partial_\nu F^{\mu\nu} = j_{\mathrm{e}}^{\mu} \, , \qquad \partial_\nu \widetilde{F}^{\mu\nu} = j_{\mathrm{m}}^{\mu} \, ,$$

where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the Hodge dual of $F_{\mu\nu}$.

These equations are invariant under the simultaneous replacements

$$F^{\mu\nu} \to \widetilde{F}^{\mu\nu} \, , \quad \widetilde{F}^{\mu\nu} \to -F^{\mu\nu} \, , \quad j_{\mathrm{e}}^{\mu} \to j_{\mathrm{m}}^{\mu} \, , \quad j_{\mathrm{m}}^{\mu} \to -j_{\mathrm{e}}^{\mu} \, .$$

Self-dual electrodynamics.

Why is EM duality of type (***B***)? Electric, magnetic charges are described by couplings q_e, q_m, which are part of the data that defines the theory, and the duality transformation exchanges these couplings as $q_e \longleftrightarrow q_m$.

In the vacuum case $j_{\mathrm{e}}^{\mu} = j_{\mathrm{m}}^{\mu} = 0$, there is no coupling data to change, so this duality transformation relates two instances of the same physical theory. Thus vacuum Maxwell electrodynamics is **duality-invariant**.

As an analogy, this is a bit like going to the self-dual radius $R = \sqrt{\alpha'}$ in the T-duality example, where the transformation $R \to \frac{\alpha'}{R}$ becomes

$$\sqrt{\alpha'} \to \frac{\alpha'}{\sqrt{\alpha'}} = \sqrt{\alpha'} \, ,$$

so the duality maps the radius to itself.

Unlike a generic duality relating different theories, a self-duality maps a theory to *itself*, and can be viewed as a form of enhanced symmetry.

Other duality invariant theories.

From now on, we restrict to the source-free case ($j_{\mathrm{e}}^{\mu} = j_{\mathrm{m}}^{\mu} = 0$). One might ask whether other theories, besides Maxwell, enjoy duality-invariance.

Given a Lagrangian $\mathcal{L}(F_{\mu\nu})$, consider a variation

$$\delta_\theta F_{\mu\nu} = \theta G_{\mu\nu}(F) \,, \qquad \widetilde{G}_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} \,, \quad G_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu\rho\tau} \widetilde{G}^{\rho\tau} \ .$$

If the Euler-Lagrange equations of $\mathcal{L}$ are invariant under δ_θ, we say that $\mathcal{L}$ defines a **duality-invariant** (or **self-dual**) **theory** [Gaillard, Zumino '81].

Another important duality-invariant theory is Born-Infeld electrodynamics, which describes the worldvolume gauge theory on a D-brane.

Likewise, given any function $\mathcal{O}(F_{\mu\nu})$ in a theory described by a Lagrangian $\mathcal{L}$, if $\delta_\theta \mathcal{O} = 0$ then $\mathcal{O}$ is said to be a **duality-invariant function**.

A curious fact about Born-Infeld.

Born-Infeld theory has an interesting property which motivated our work. If

$$\mathcal{L}_{\text{BI}} = \frac{1}{\alpha^2} \left(1 - \sqrt{1 + \frac{\alpha^2}{2} F_{\mu\nu} F^{\mu\nu} - \frac{\alpha^4}{16} \left(F_{\mu\nu} \widetilde{F}^{\mu\nu} \right)^2 } \right)$$

is the Born-Infeld Lagrangian and

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}_{\text{BI}}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_{\text{BI}} \, ,$$

is its energy-momentum tensor, then the theory obeys an equation

$$\frac{\partial \mathcal{L}_{\text{BI}}}{\partial \alpha^2} = \frac{1}{8} \left(T^{\mu\nu} T_{\mu\nu} - \frac{1}{2} \left(T^{\mu}{}_{\mu} \right)^2 \right) \, ,$$

$$\lim_{\alpha \to 0} \mathcal{L}_{\text{BI}} = \mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \, ,$$

where α is related to the brane tension [Conti, Iannella, Negro, Tateo '18].

Thus a family of duality-invariant theories is obtained by beginning with the Maxwell theory and deforming it by a function of the stress tensor.

Roadmap.

Our paper addresses the question:

Is there a more general relationship between duality-invariance and differential equations involving the stress tensor?

We find that the answer is yes, and the deeper relationship can be understood either from a direct analysis of the Lagrangian or using the auxiliary field formulation of [Ivanov, Zupnik '03].

In the rest of the talk, I hope to explain this in more detail:

- ☑ Part 1: Introduction.
- ☐ Part 2: Some theorems about duality invariance.
- ☐ Part 3: Auxiliary field formulation.
- ☐ Part 4: Summary and future directions.

Part 2: Some theorems about duality invariance.

Parametrizing a general Lagrangian.

We are interested in Lagrangians $\mathcal{L}(F)$ which depend on a single Abelian field strength $F_{\mu\nu}$, but not higher derivative terms like $\partial_\rho F_{\mu\nu}$.

The most general Lorentz scalar that can be constructed from a 2-form in 4 dimensions is a function of two variables.* We can therefore choose

$$\mathcal{L} = \mathcal{L}(S, P)\,, \qquad S = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad P = -\frac{1}{4} F_{\mu\nu} \widetilde{F}^{\mu\nu}\,,$$

where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\tau} F_{\lambda\tau}$ as before.

The theory defined by such a Lagrangian will be duality-invariant if

$$\mathcal{L}_S^2 - \frac{2S}{P}\mathcal{L}_S \mathcal{L}_P - \mathcal{L}_P^2 = 1\,.$$

*One can understand this by noting that a skew-symmetric 4×4 matrix has 4 imaginary eigenvalues, which come in complex-conjugate pairs $\pm i\lambda_1$ and $\pm i\lambda_2$.

Infinitesimal perturbation.

Suppose that we deform a duality-invariant Lagrangian $\mathcal{L}$ as

$$\mathcal{L}(S, P) \longrightarrow \mathcal{L}'(S, P) = \mathcal{L}(S, P) + \lambda \mathcal{O}(S, P) \,,$$

where λ does not transform under duality rotations.

The condition for the new theory defined by $\mathcal{L}'$ to remain duality invariant, at leading order in λ, is

$$\mathcal{L}_S \mathcal{O}_S - \frac{S}{P} \left(\mathcal{L}_P \mathcal{O}_S + \mathcal{L}_S \mathcal{O}_P \right) - \mathcal{L}_P \mathcal{O}_P = 0 \,.$$

But this is precisely the same equation that arises from imposing

$$\delta_\theta \mathcal{O} = 0 \,,$$

in the original theory described by $\mathcal{L}$. Thus duality-invariant functions correspond with infinitesimal duality-preserving deformations.

Flows from duality-invariant functions.

The preceding slide is the main idea of our first simple result.

Theorem 1

Consider a family of theories satisfying the differential equation

$$\frac{\partial \mathcal{L}^{(\lambda)}(S,P)}{\partial \lambda} = \mathcal{O}^{(\lambda)}(S,P) \, , \qquad \mathcal{L}^{(0)}(S,P) = \mathcal{L}(S,P) \, ,$$

with $\mathcal{O}^{(\lambda)}$ being a duality-invariant function, $\delta_\theta^{(\lambda)} \mathcal{O}^{(\lambda)} = 0$.

If the Lagrangian $\mathcal{L}(S,P)$ describes a duality-invariant theory, then all theories associated with the Lagrangians $\mathcal{L}^{(\lambda)}(S,P)$ are duality invariant.

Note that the functions $\mathcal{O}^{(\lambda)}$ must be invariant with respect to the duality transformation in the theory with parameter λ, since $\delta_\theta^{(\lambda)}$ depends on $\mathcal{L}^{(\lambda)}$.

Dependence of duality-invariant functions.

We have reduced the task of describing duality-preserving deformations to that of enumerating all duality-invariant functions.

It turns out that all such functions are dependent upon one another.

Theorem 2

Given a duality-invariant theory with Lagrangian $\mathcal{L}(S, P)$, any two duality-invariant functions $f(S, P)$ and $g(S, P)$ are functionally dependent.

Once we have found one such non-trivial function – say, one built from the stress tensor like $T^{\mu\nu} T_{\mu\nu}$ – all others can be obtained as functions of it.

Proof of theorem 2.

Proof.

First we recall that $f(S,P)$ is duality invariant if and only if

$$(S\mathcal{L}_P - P\mathcal{L}_S)\, f_S + (S\mathcal{L}_S + P\mathcal{L}_P)\, f_P = 0\,.$$

To study this condition, it is convenient to introduce a vector field

$$\vec{v}(S,P) = v^S \partial_S + v^P \partial_P = (S\mathcal{L}_P - P\mathcal{L}_S)\, \partial_S + (S\mathcal{L}_S + P\mathcal{L}_P)\, \partial_P\,.$$

This vector field is non-vanishing, which we show by contradiction.

If $\vec{v}(S,P) = 0$, we would have $S\mathcal{L}_P - P\mathcal{L}_S = 0$ and $S\mathcal{L}_S + P\mathcal{L}_P = 0$. It follows that $\mathcal{L}(S,P) = L\left(S^2 + P^2\right)$, for some function $L(x)$ of a single variable and that $L(x)$ is a homogeneous function of degree 0, and therefore $\mathcal{L} = \mathrm{const}$, which is a contradiction.

Proof of theorem 2.

Proof.

The duality invariance condition tells us that the vector field

$$\vec{f}(S, P) = f_S \partial_S + f_P \partial_P \,,$$

is orthogonal to $\vec{v}(S, P)$,

$$v^S f_S + v^P f_P = 0 \,.$$

Now consider another duality-invariant function $g(S, P)$,

$$v^S g_S + v^P g_P = 0 \,,$$

and similarly define a vector field $\vec{g}$ by

$$\vec{g}(S, P) = g_S \partial_S + g_P \partial_P \,.$$

Proof of theorem 2.

Proof.

Since both $\vec{f}$ and $\vec{g}$ are non-zero and orthogonal to $\vec{v}$, the two must be parallel, i.e. $\vec{f} = h(S, P)\vec{g}$ for some function $h(S, P)$, and therefore

$$\det \left(\begin{bmatrix} f_S & f_P \\ g_S & g_P \end{bmatrix} \right) = 0 \, .$$

We conclude that f and g are functionally dependent,

$$\Upsilon(f, g) = 0 \, ,$$

for some function of two variables Υ.

□

Duality-invariance of stress tensor.

Since all duality-invariant functions are dependent, it suffices to understand one example of such a function.

One physically motivated, and easy to compute, example is the energy-momentum tensor. One can show by direct computation* that

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \mathcal{L}_S F_{\mu}{}^{\rho} F_{\nu\rho} + \mathcal{L}_P F_{(\mu}{}^{\rho} \widetilde{F}_{\nu)\rho} + g_{\mu\nu} \mathcal{L}$$

is duality invariant, as is well-known. We wish to deform the Lagrangian by Lorentz scalars, so it is natural to use scalar combinations built from $T_{\mu\nu}$,

$$\begin{aligned} T^{\mu}{}_{\mu} &= -4 \left(S \mathcal{L}_S + P \mathcal{L}_P - \mathcal{L} \right) \, , \\ T^{\mu\nu} T_{\mu\nu} &= 4 \left(\left(\mathcal{L} - P \mathcal{L}_P \right)^2 - 2 S \mathcal{L}_S \left(\mathcal{L} - P \mathcal{L}_P \right) + (P^2 + 2 S^2) \mathcal{L}_S^2 \right) \, . \end{aligned}$$

*Alternatively, one may appeal to the fact that the derivative of the Lagrangian with respect to a duality-invariant quantity (like the metric) is itself duality-invariant.

Overview of results.

Our main conclusion is a consequence of the following three facts:

1. A family of theories with Lagrangians $\mathcal{L}(\lambda)$ are self-dual if and only if $\partial_\lambda \mathcal{L}(\lambda) = \mathcal{O}(\lambda)$, where $\mathcal{O}(\lambda)$ is a duality-invariant scalar function.
2. Any two duality-invariant functions $\mathcal{O}_1$, $\mathcal{O}_2$ are functionally dependent. That is, $\Upsilon(\mathcal{O}_1, \mathcal{O}_2) = 0$ for some function Υ.
3. Any Lorentz scalar constructed from the stress tensor $T_{\mu\nu}$, such as $T^{\mu\nu} T_{\mu\nu}$ and $T^{\mu}{}_{\mu}$, is a duality-invariant function.

It follows that any family of duality-invariant Lagrangians obeys a flow

$$\frac{\partial \mathcal{L}}{\partial \lambda} = f\left(T_{\mu\nu} \right) \, ,$$

for some function f. Conversely, given any such function f and initial condition, we generate a family of duality-invariant theories.

Part 3: Auxiliary field formulation.

Introducing auxiliary fields.

Our intuition from the preceding section is that description of duality-invariant electrodynamics is "secretly" a univariate problem.

One way to make this reduction manifest is to use the formulation of [Ivanov, Zupnik '03]. To do this, we introduce an auxiliary 2-form field $V_{\mu\nu} = -V_{\nu\mu}$, and convert both $F_{\mu\nu}$ and $V_{\mu\nu}$ to spinor indices as

$$F_\alpha{}^\beta = -\frac{1}{4}(\sigma^\mu)_{\alpha\dot\beta}(\tilde\sigma^\nu)^{\dot\beta\beta}F_{\mu\nu}\,, \quad \bar F_{\dot\alpha}{}^{\dot\beta} = \frac{1}{4}(\tilde\sigma^\mu)^{\dot\beta\beta}(\sigma^\nu)_{\beta\dot\alpha}F_{\mu\nu}\,,$$

$$V_\alpha{}^\beta = -\frac{1}{4}(\sigma^\mu)_{\alpha\dot\beta}(\tilde\sigma^\nu)^{\dot\beta\beta}V_{\mu\nu}\,, \quad \bar V_{\dot\alpha}{}^{\dot\beta} = \frac{1}{4}(\tilde\sigma^\mu)^{\dot\beta\beta}(\sigma^\nu)_{\beta\dot\alpha}V_{\mu\nu}\,,$$

where the σ^μ, $\tilde\sigma^\mu$ are Pauli matrices. We also define the scalars

$$\varphi = F^{\alpha\beta}F_{\alpha\beta}\,, \qquad \bar\varphi = \bar F_{\dot\alpha\dot\beta}\bar F^{\dot\alpha\dot\beta}\,,$$

$$\nu = V^{\alpha\beta}V_{\alpha\beta}\,, \qquad \bar\nu = \bar V_{\dot\alpha\dot\beta}\bar V^{\dot\alpha\dot\beta}\,,$$

$$V\cdot F = V^{\alpha\beta}F_{\alpha\beta}\,, \qquad \bar V\cdot\bar F = \bar V_{\dot\alpha\dot\beta}\bar F^{\dot\alpha\dot\beta}\,.$$

A family of auxiliary field models.

Using this notation, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\varphi + \overline{\varphi}\right) + \nu + \bar{\nu} - 2\left(V \cdot F + \overline{V} \cdot \overline{F}\right) + E\left(\nu\bar{\nu}\right) \, ,$$

which depends on a function $E(a)$ of one real variable $a = \nu\bar{\nu}$.

Claim: after integrating out the auxiliary fields V, $\overline{V}$, any such Lagrangian gives rise to a self-dual theory of electrodynamics.

In fact, we can see how the duality transformation acts on V and F:

$$\delta_\theta \begin{pmatrix} V_{\alpha\beta} \\ F_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} -i\theta V_{\alpha\beta} \\ i\theta(F_{\alpha\beta} - 2V_{\alpha\beta}) \end{pmatrix} \, ,$$

with a similar action on $\overline{V}$, $\overline{F}$. Thus retaining the auxiliaries V, $\overline{V}$ has turned the duality action into a linear transformation of the fields.

Stress tensor flows with auxiliary fields.

For the previous model, one can compute the Lorentz scalars

$$
\begin{aligned}
T^{\mu}{}_{\mu} &= 4\left(E - 2aE_a\right) \, , \\
T^{\mu\nu}T_{\mu\nu} &= 4\left(E^2 + a\left(\left(1 + aE_a^2\right)^2 - 4EE_a\right)\right) \, .
\end{aligned}
$$

Although these two quantities are functionally dependent, as they must be, it is convenient to use both of them in defining flow equations.

Suppose we now deform the previous Lagrangian as

$$
\frac{\partial \mathcal{L}}{\partial \lambda} = f\left(T^{\mu\nu}T_{\mu\nu}, T^{\mu}{}_{\mu}\right) \, .
$$

Since the right side can be expressed purely in terms of E, a, and E_a, this gives rise to a closed differential equation for the function $E(a)$.

In particular, such a deformation leaves us within the *same* family of auxiliary field models, and merely modifies the interaction function $E(a)$.

An example marginal flow.

Example. Consider a flow driven by the so-called "root-$T\overline{T}$" operator,

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \frac{1}{2}\sqrt{T^{\mu\nu} T_{\mu\nu} - \frac{1}{4}\left(T^{\mu}{}_{\mu}\right)^2} \, .$$

This combination has mass dimension 4 and is thus classically marginal. If we begin from the initial condition $E(a) = 0$, which corresponds to the Maxwell theory, the differential equation reduces to

$$\frac{\partial E}{\partial \gamma} = \sqrt{a}\left(1 - a E_a^2\right) \, ,$$

with solution

$$E(a, \gamma) = 2 \tanh\left(\frac{\gamma}{2}\right)\sqrt{a} \, .$$

What theory of electrodynamics does this describe?

The Modified Maxwell theory emerges.

After integrating out the auxiliaries, the preceding interaction function gives rise to the theory

$$\mathcal{L}_{\text{ModMax}}(\gamma) = -\frac{1}{4}\left(\cosh(\gamma) F_{\mu\nu}F^{\mu\nu} + \sinh(\gamma)\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + \left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)^2}\right) \,,$$

which is the "ModMax theory" of [Bandos, Lechner, Sorokin, Townsend, '20].

This is the unique conformally-invariant and duality-invariant extension of the Maxwell theory. It was already known that this theory obeys a root-$T\overline{T}$ flow in this conventional presentation (without auxiliary fields).

There is a generalization to a two-parameter family $\mathcal{L}_{\text{ModMax-BI}}(\gamma, \alpha)$ of theories which reduces to $\mathcal{L}_{\text{ModMax}}(\gamma)$ when $\alpha \to 0$ and to the Born-Infeld Lagrangian $\mathcal{L}_{\text{BI}}(\alpha)$ when $\gamma = 0$ [Bandos, Lechner, Sorokin, Townsend, '20].

The ModMax-Born-Infeld theory.

The Lagrangian for this duality-invariant two parameter family is

$$\mathcal{L}_{\text{ModMax-BI}}(\gamma,\alpha) = \frac{1}{\alpha^2}\Bigg[1 - \Bigg(1 + \frac{\alpha^2}{2}\Bigg(\cosh(\gamma) F_{\mu\nu}F^{\mu\nu}$$
$$+ \sinh(\gamma)\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + \left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)^2}\Bigg) - \frac{\alpha^4}{16}\left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)^2\Bigg)^{1/2}\Bigg],$$

and the theory satisfies two commuting stress tensors flows,

$$\begin{array}{ccc} \mathcal{L}_{\text{Maxwell}} & \xrightarrow{\quad\mathcal{R}\quad} & \mathcal{L}_{\text{ModMax}}(\gamma) \\ \Big\downarrow{\scriptstyle \mathcal{O}_{T\overline{T}}} & & \Big\downarrow{\scriptstyle \mathcal{O}_{T\overline{T}}} \\ \mathcal{L}_{\text{BI}}(\alpha) & \xrightarrow[\quad\mathcal{R}\quad]{} & \mathcal{L}_{\text{ModMax-BI}}(\gamma,\alpha) \end{array} ,$$

where $\mathcal{O}_{T\overline{T}} = \frac{1}{8}\left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}\left(T^{\mu}{}_{\mu}\right)^2\right)$ and $\mathcal{R} = \frac{1}{2}\sqrt{T^{\mu\nu}T_{\mu\nu} - \frac{1}{4}\left(T^{\mu}{}_{\mu}\right)^2}$.

An auxiliary field representation.

Another result of our paper is to find a new auxiliary field form of $\mathcal{L}_{\text{ModMax-BI}}(\gamma, \alpha)$. Instead of the formulation involving $E(a)$, define

$$\begin{gathered}\mathcal{L} = \frac{\varphi(\mu - 1)}{2(1+\mu)} + \frac{\bar{\varphi}(\bar{\mu} - 1)}{2(1+\bar{\mu})} + H(\mu, \bar{\mu}) \,, \\ H(\mu, \bar{\mu}) = E(\nu, \bar{\nu}) - \nu\mu - \bar{\nu}\bar{\mu} \,, \\ \mu(\nu, \bar{\nu}) = E_{\nu} \;, \quad \bar{\mu}(\nu, \bar{\nu}) = E_{\bar{\nu}} \,.\end{gathered}$$

The ModMax-Born-Infeld theory corresponds to the interaction function

$$H_{\gamma \text{BI}}(b; \gamma) = \frac{1}{\alpha^2} \frac{b - 1 + (1+b)\cosh(\gamma) + 2\sqrt{b}\sinh(\gamma)}{b - 1} \,,$$

where $b = \mu\bar{\mu}$. Like before, any stress tensor deformation of a theory in this presentation simply changes the interaction function $H(b)$, while remaining within the class of duality-invariant models.

Part 4: Summary and future directions.

Duality-invariant families = stress tensor flows.

To summarize some of our key results,

1. We showed that every duality-invariant family of electrodynamics satisfies a flow equation sourced by a function of the stress tensor.
2. Conversely, any function of the stress tensor can be used to generate a line of self-dual electrodynamics theories from an initial condition.
3. We studied these stress tensor flows using the Ivanov-Zupnik auxiliary field formulation, which makes certain aspects manifest.
4. We derived a new auxiliary field representation of the two-parameter ModMax-Born-Infeld theory, which satisfies two commuting flows.

Future directions.

Some interesting areas for further investigation:

1. The same PDE which is satisfied by the Lagrangian $\mathcal{L}(S, P)$ of a duality-invariant theory also occurs in the study of $2d$ integrable sigma models. Can one use an auxiliary field formulation in that setting?

2. Can one perform a similar analysis to explore deformations of $6d$ theories of a two-form $B_{\mu\nu}$ with self-dual 3-form field strength $H_{\mu\nu\rho}$?*

3. Stress tensor deformations in two spacetime dimensions, like the $T\overline{T}$ deformation, can be interpreted geometrically with a field-dependent metric. Can one use this to learn about theories of electrodynamics?

Answering some of these questions could provide new insights about how to use flow equations to explore the space of field theories.

*Work in progress with Sergei Kuzenko, Kurt Lechner, Dmitri Sorokin, Gabriele Tartaglino-Mazzucchelli.

Thank you for your a$T\overline{T}$ention!

Backup: Comments on Relation to $T\overline{T}$.

Quantum deformations.

In my talk, we only discussed classical deformations of the action. Such deformations are not guaranteed to be well-defined in the quantum theory.

A famous stress tensor deformation which *does exist* quantum mechanically is the $T\overline{T}$ deformation of two-dimensional quantum field theories.

In any translation-invariant $2d$ QFT there is an operator

$$\mathcal{O}_{T\overline{T}}(x) = \lim_{y \to x} \left(T^{\mu\nu}(x) T_{\mu\nu}(y) - T^{\mu}{}_{\mu}(x) T^{\nu}{}_{\nu}(y) \right) .$$

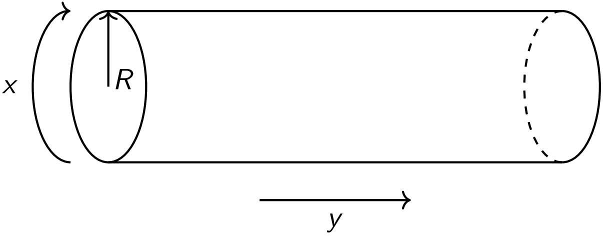
Despite involving a coincident-point limit of local operators, this point-splitting procedure gives a well-defined result [Zamolodchikov 2004].

One can therefore deform any translation-invariant $2d$ QFT by this operator $\mathcal{O}_{T\overline{T}}$, even at the quantum level.

Solvability of $T\overline{T}$.

The $T\overline{T}$ deformation is **solvable** in the sense that observables like the spectrum, torus partition function, and S-matrix of a $T\overline{T}$-deformed theory can be expressed in terms of quantities in the undeformed (seed) theory.

As an example, consider the spectrum of energies $E_n(R)$ for a $2d$ QFT on a cylinder of radius R:



Flow equation for energies.

Suppose that we deform the theory by

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2} \int d^2 x \, \left(T^{(\lambda)}_{\mu\nu} T^{(\lambda)\mu\nu} - \left(T^{(\lambda)\mu}{}_{\mu} \right)^2 \right) .$$

Using the expressions

$$T_{yy} = -\frac{1}{R} E_n(R) \, , \qquad T_{xx} = -\frac{\partial E_n(R)}{\partial R} \, , \qquad T_{xy} = \frac{i}{R} P_n(R) \, ,$$

for stress tensor components, one finds that the spectrum flows according to the inviscid Burgers' equation,

$$\frac{\partial E_n}{\partial \lambda} = E_n \frac{\partial E_n}{\partial R} + \frac{P_n^2}{R} \, ,$$

as explained in [Cavaglià, Negro, Szécsényi, Tateo '16].

Connections to string theory.

One way to see that $T\overline{T}$ is related to string theory is to solve the flow equation for the Lagrangian beginning from a seed theory

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi \,,$$

which gives

$$\mathcal{L}_\lambda = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 2\lambda \partial^\mu \phi \partial_\mu \phi}\right) \,.$$

This is the Lagrangian for a static gauge Nambu-Goto string with a three-dimensional target space [Cavaglià, Negro, Szécsényi, Tateo '16].

The ability to find a closed-form expression for the deformed Lagrangian is another incarnation of solvability.

This calculation is very similar to the one which shows that our $T\overline{T}$-like deformation of the $4d$ Maxwell theory yields the Born-Infeld theory.